The "Cameo Principle" and the Origin of Scale-Free Graphs in Social Networks[1]

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We formulate a simple edge generation rule based on an inverse like mass action principle for random graphs over a structured vertex set. We show that under very weak assumptions on this structure one obtains a scale free distribution for the degree. We furthermore introduce and study a "my friends are your friends" local search principle which makes the clustering coefficient large.

1. INTRODUCTION

Random graphs with a scale free distribution for the degree seem to appear very frequently in a great variety of real life situations like the World-WideWeb, the Internet, social networks, linguistic networks, citation networks and biochemical networks. An excellent recent survey is the article by Albert and Barabasi.[1]

Despite the large number of articles dealing with specific properties of such networks little is known about origin and formation principles of such graphs in the general framework. Most popular became the model of Albert and Barabasi[2] where graphs grow by successively adding a new vertex say x at each time step and forming edges between this new vertex and the already existing ones by assuming that the probability that x forms an edge with a vertex y whose degree is k is about $const \cdot \frac{k}{N}$ where N is the total number of vertices at that time. This gives a scale free graph although

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a very homogeneous one with little room for incorporating additional structures. The principle used in this construction, and the many variants of it, could be seen as a degree-mass-action principle since the degree acts here as a kind of positive affinity parameter for alignment of new vertices. Of course knowledge about the degree of a vertex is in real networks seldom available for other vertices. For instance in the social networks of friendships an individual certainly does not make the formation of a new friendship dependent of checking how many friends the potential new candidate already has (not to mention that it would be rather difficult to estimate this number). So we seek for an edge formation principle more related to a pre-given structure on the vertex set. We assume that this structure is specified by a real positive parameter $\omega \in \mathbb{R}^+$—like social importance, money or beauty—with a given probability distribution $\varphi(\omega)$. Furthermore we think of an edge between vertices x and y as the result of a directed choice made by either x or y (symbolized by $x \to y$ or $y \to x$). This seems to be very reasonable since in many real life networks edges are formed that way. Note that although the edge creation is a directed process we consider the resulting graph as an undirected one since for the majority of relevant transmission processes on the network the original orientation of an edge is irrelevant. The crucial assumption about the pairing probability is now a kind of inverse mass-action-principle. Namely that the probability that a vertex x which makes a choice chooses a given vertex y with parameter $\omega(y)$ is proportional to $[\varphi(\omega(y))]^{-\alpha} \cdot \frac{1}{N}$ with some $\alpha > 0$ and N being the total number of vertices. Due to this principle it is not the actual value ω of a vertex y which is relevant for the pairing probability but rather it's relative frequency of appearance in the "population." This is in a certain sense a market rule—the more rare a property is, the more attractive it becomes for others. Of course the property associated with ω has to be positive in perception. We think that this "Ansatz" captures the essence of many real life network formation processes.

Instead of using a technical name for the principle like "inverse mass action" we want to call it the "Cameo-principle" having in mind the attractiveness, rareness and beauty of the small medallion with a profiled head in relief called Cameo. And it is exactly their rareness and beauty which gives them their high value.

Our basic assumptions can be summarized as follows:

• The parameter ω is independent identically distributed (i.i.d.) over the vertex set with a smooth monotone decreasing density function $\varphi(\omega)$.

• Edges are formed by a sequence of *choices*. By a *choice* we mean that a vertex x chooses another vertex, say y, to form an edge between y and x.

A vertex can make several choices. All choices are assumed to be independent of each other.

• If x makes a choice the probability of choosing y depends only on the relative density of $\omega(y)$ and is of the form

$$\Pr\{x \to y \,|\, \omega(y)\} \sim [\varphi(\omega(y))]^{-\alpha} \cdot \frac{1}{N}; \qquad \alpha \in (0; 1). \qquad (1)$$

• A pre-given outdegree distribution determines the number of choices made by the vertices. The total number of choices (and therefore the number of edges) is assumed to be about $\text{const} \cdot N$.

The striking observation under the above assumptions is now the emergence of a scale-free degree distribution independent of the choice of the ω-distribution. Furthermore it can be shown that the exponent in the degree distribution becomes independent of $\varphi(\omega)$ if the tail of φ decays faster then any power law. The precise formulation of the above statements is given in Theorem 1 and 2 in the next section.

2. ANALYSIS OF THE MODEL

Let $V_N = \{1, \ldots, N\}$ be the vertex set of a random graph space. We are mainly interested in the asymptotic properties for N being very large. We assign i.i.d. to each element x from the set V_N a continuous positive real random variable (r.v.) $\omega(x)$ taken from a distribution with density function $\varphi(\omega)$. The variable ω can be interpreted as a parametrization of V_N. For a set $C_{\omega_0, \omega_1} = \{x : \omega(x) \in [\omega_0, \omega_1]\}$ we obtain $\mathbb{E}(\sharp C_{\omega_0, \omega_1}) = N \cdot \int_{\omega_0}^{\omega_1} \varphi(\omega)\, d\omega$. Without loss of generality we always assume $\varphi > 0$ on $[0, \infty)$. As a technical assumption we will furthermore need the following:

Assumption 1. $\varphi \in C^2([0, \infty))$ and the second derivatives $D^2(\varphi^\mu)$ have no zeros for $|\mu| \in (0, \mu_0)$ and $\omega > \omega_0(\mu)$ (this is just a monotonicity assumption on the tail of φ).

Edges are created by a directed process in which the basic events are choices made by the vertices. All choices are assumed to be i.i.d. The number of times a vertex x makes a choice is itself a random variable which may depend on x. We call this r.v. $d_{\text{out}}(x)$. The number of times a vertex x was chosen in the edge formation process is called the indegree $d_{\text{in}}(x)$. Each choice generates a directed edge. We are mostly interested in the corresponding undirected graph. If we speak in the following about outdegree and indegree we refer just to the original direction in the edge formation process. Let $p_\omega = \Pr\{x \to y \,|\, \omega = \omega(y)\}$ be the basic probability that a

vertex y with a fixed value of ω is chosen by x if x is about to make a choice. For a given realization ξ of the r.v. ω over V_N we assume:

$$p_\omega(\xi, N) = \frac{A(\xi, N)}{[\varphi(\omega)]^\alpha} \cdot \frac{1}{N} \qquad (2)$$

where $\alpha \in (0, 1)$ and $A(\xi, N)$ is a normalization constant. It is easy to see that the condition $\int_0^\infty [\varphi(\omega)]^{1-\alpha} \, d\omega < \infty$ is necessary and sufficient to get $A(\xi, N) \to A > 0$ for $N \to \infty$ where convergence is in the sense of probability. Therefore we need $\alpha < 1$. One might argue that the choice probabilities should depend more explicitly on the actual realization ξ of the r.v. ω over V_N—not only via the normalization constant. The reason not to do so is twofold. First it is mathematically unpleasant to work with the empirical distribution of ω induced by the realization ξ since one had to use a somehow artificial N-dependent coarse graining. Second the empirical distribution is not really "observed" by the vertices (having in mind for instance individuals in a social network). What seems to be relevant is more the common believe about the distribution of ω. In this sense our setting is a natural one. To keep the analysis simple we will first assume in Section 2.1. that the r.v. $d_{out}(x)$ is constant and equal to k_0 for all x. We will discuss the situation of a variable d_{out} in Section 2.2. Since we want to show that under the above conditions on φ we obtain a power law distribution for the degree we have to say a few words about the notion of power-law distribution in our framework. We say that a discrete probability distribution ψ on $\mathbb{N}$ is of power law type with exponent β if

$$\psi(k) = \frac{1}{k^{\beta + o(1)}} \qquad \text{for} \quad k > 1. \qquad (3)$$

This means that our concept of power law distribution is an asymptotic one. We think that this is a very reasonable notion since in any empirical observation one can see only an approximation to a power law.

Before starting with the proof of our claims we would like to mention that the emergence of a power law distribution in our setting is not so surprising as it might seem at the first look. The situation is best explained in form of an example. Let $\varphi(\omega) = C \cdot e^{-\omega}$ and define a new variable $\omega^* = \frac{1}{[\varphi(\omega)]^\alpha} = \frac{e^{\omega\alpha}}{C^\alpha}$. The new variable ω^* can be seen as the effective parameter to which the vertex choice process applies. What is the induced distribution of ω^*? With $F(z) = \Pr\{\omega^* < z\}$ we obtain

$$F(z) = \int_0^{\frac{1}{\alpha} \ln C^\alpha \cdot z} \varphi(\omega) \, d\omega = -z^{\frac{-1}{\alpha}} - C \qquad (4)$$

and therefore the ω^*-distribution $\phi(\omega^*) = \frac{1}{\alpha} \cdot \frac{1}{(\omega^*)^{1+(1/\alpha)}}$. This is a power law distribution with an exponent depending only on α.

2.1. The Case of Constant Outdegree ($d_{out}(x) = k_0$)

The main result of the paper is contained in the following

Theorem 1. (i)   Let $\varphi(\omega) = \frac{1}{\omega^{\beta + o(1)}}$ with $\beta > 2$.
Then $\lim_{N \to \infty} \Pr\{d(x) = k \mid x \in V_N\} = \frac{1}{k^{1+(1/\alpha)-(1/\alpha\beta)+o(1)}}$.

(ii)   Let $\varphi(\omega)$ fulfill the Assumption 1.
Then $\lim_{N \to \infty} \Pr\{d(x) = k \mid x \in V_N\} = \frac{1}{k^{1+(1/\alpha)+o(1)}}$.

Remark.   The theorem says, that under Assumption 1 the degree distributions for the finite random graphs of our model converge for $N \to \infty$ to a power law distribution in the sense of formula (3).

Proof.   Since by assumption the outdegree is constant and $d(x) = d_{in}(x) + d_{out}(x) = d_{in}(x) + k_0$ asymptotically almost surely, we have to consider only the indegree distribution. First we compute the expectation (in the limit $N \to \infty$):

$$\mathbb{E}(d_{in}(x) \mid \omega = \omega(x)) = \frac{k_0 \cdot A}{[\varphi(\omega)]^\alpha} \tag{5}$$

and therefore

$$\mathbb{E}(d_{in}(x)) = k_0 \cdot A \cdot \int [\varphi(\omega)]^{1-\alpha} \, d\omega. \tag{6}$$

For a x with fixed $\omega = \omega(x)$ we get for the indegree distribution $\psi_\omega(k) = \Pr\{d_{in}(x) = k \mid \omega = \omega(x)\}$ a binomial distribution (here we count also multiple edges but the probability for their appearance vanishes for large N)

$$\psi_\omega(k) = \binom{k_0 N}{k} \left(\frac{A(\xi, N)}{[\varphi(\omega)]^\alpha N} \right)^k \left(1 - \frac{A(\xi, N)}{[\varphi(\omega)]^\alpha N} \right)^{k_0 N - k} \tag{7}$$

which converges in the limit of large N to a Poisson distribution:

$$\psi_\omega(k) = \frac{c_\omega^k}{k!} e^{-c_\omega} \tag{8}$$

with $c_\omega = \frac{k_0 \cdot A}{[\varphi(\omega)]^\alpha}$. Finally we obtain for the unconditional limiting probability $\psi(k) = \lim_{N \to \infty} \Pr(d_{in}(x) = k)$ the expression

$$\psi(k) = \int \varphi(\omega) \frac{c_\omega^k}{k!} e^{-c_\omega} \, d\omega. \tag{9}$$

The main contribution to $\psi(k)$ comes from a rather small interval of ω-values—called $I_{ess}(k)$—with the property that for $\omega_0 \in I_{ess}(k)$ the expectation $\mathbb{E}(d_{in}(x) \mid \omega_0 = \omega(x))$ is of order k. The rapid decay of the Poisson distribution will guarantee that the remaining parts of the integral become arbitrary small for large k. To fill in the details let

$$I_{ess}(k, \gamma) = \left[\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k - k^{\frac{1}{2}+\gamma}} \right)^{\frac{1}{\alpha}} \right), \varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k + k^{\frac{1}{2}+\gamma}} \right)^{\frac{1}{\alpha}} \right) \right] \tag{10}$$

with $0 < \gamma \ll 1$. To estimate $\psi_{ess}(k) = \int_{I_{ess}(k, \gamma)} \varphi(\omega) \frac{c_\omega^k}{k!} e^{-c_\omega} \, d\omega$ we define a new variable ε in the interval $[-k^{\frac{1}{2}+\gamma}, k^{\frac{1}{2}+\gamma}]$ by $\omega = \varphi^{-1}((\frac{k_0 \cdot A}{k+\varepsilon})^{\frac{1}{\alpha}})$ and transform the integral into

$$\psi_{ess}(k) = \int_{-k^{\frac{1}{2}+\gamma}}^{k^{\frac{1}{2}+\gamma}} \frac{-(k_0 \cdot A)^{\frac{1}{\alpha}}}{\alpha(k+\varepsilon)^{1+\frac{1}{\alpha}}} \cdot \frac{\varphi \circ \varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon} \right)^{\frac{1}{\alpha}} \right)}{(D\varphi)\left(\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon} \right)^{\frac{1}{\alpha}} \right) \right)} \cdot \frac{\tilde{c}_\varepsilon^k}{k!} e^{-\tilde{c}_\varepsilon} \, d\varepsilon \tag{11}$$

with

$$\tilde{c}_\varepsilon = \frac{k_0 \cdot A}{\left[\varphi\left(\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon} \right)^{\frac{1}{\alpha}} \right) \right) \right]^\alpha} = k + \varepsilon. \tag{12}$$

Here we used the relation $D\varphi^{-1}(x) = [(D\varphi)(\varphi^{-1}(x))]^{-1}$. Using Stirling's formula $k! = (2\pi k)^{\frac{1}{2}} (\frac{k}{e})^k (1+o(1))$ and the approximation $(1+\frac{\varepsilon}{k})^k = e^{\varepsilon - \frac{\varepsilon^2}{2k} + o(1)}$ for $|\varepsilon| \ll k^{\frac{2}{3}}$ we get for the Poisson term in the integral

$$\frac{\tilde{c}_\varepsilon^k}{k!} e^{-\tilde{c}_\varepsilon} = (2\pi k)^{-\frac{1}{2}} \cdot e^{-\frac{\varepsilon^2}{2k} + o(1)}. \tag{13}$$

Since $|\varepsilon| \ll k$ we obtain from formula (11)

$$\psi_{ess}(k) = \frac{(k_0 \cdot A)^{\frac{1}{\alpha}}(1+o(1))}{\alpha k^{1+\frac{1}{\alpha}}} \cdot \int_{-k^{1/2+\gamma}}^{k^{1/2+\gamma}} \frac{-\varphi \circ \varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon} \right)^{\frac{1}{\alpha}} \right)}{(D\varphi)\left(\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon} \right)^{\frac{1}{\alpha}} \right) \right)} \cdot \frac{e^{-\frac{\varepsilon^2}{2k} + o(1)}}{(2\pi k)^{\frac{1}{2}}} \, d\varepsilon. \tag{14}$$

If the decay of φ is faster then any power-law we will show that

$$\frac{-\varphi \circ \varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right)}{(D\varphi)\left(\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right)\right)} = (k+\varepsilon)^{o(1)} \tag{15}$$

from which we can conclude that $\psi_{\text{ess}}(k) = \frac{1}{k^{1+(1/\alpha)+o(1)}}$ holds. But first we will deal with the simpler case when φ is itself of power-law type. Let us assume that φ has an exact power-law tail of the form $\frac{C}{\omega^{\beta}}$. Therefore $\varphi^{-1}(\omega) = (\frac{C}{\omega})^{\frac{1}{\beta}}$ and

$$(D\varphi)\,\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right) = \frac{-\beta C^{\frac{-1}{\beta}}(k_0 \cdot A)^{\frac{\beta+1}{\alpha\beta}}}{(k+\varepsilon)^{\frac{\beta+1}{\alpha\beta}}} \tag{16}$$

and finally

$$\frac{-\varphi \circ \varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right)}{(D\varphi)\left(\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right)\right)} = \frac{C^{\frac{1}{\beta}}(Ak_0)^{\frac{-1}{\alpha\beta}}}{\beta(k+\varepsilon)^{\frac{-1}{\alpha\beta}}}. \tag{17}$$

Since

$$\int_{-k^{1/2+\gamma}}^{k^{1/2+\gamma}} \frac{e^{-\frac{\varepsilon^2}{2k}+o(1)}}{(2\pi k)^{\frac{1}{2}}}\, d\varepsilon \simeq 1 + o(1) \tag{18}$$

we get for the essential part of the degree distribution

$$\psi_{\text{ess}}(k) = \frac{C^{\frac{1}{\beta}}(k_0 \cdot A)^{\frac{1}{\alpha}-\frac{1}{\alpha\beta}}(1+o(1))}{\beta\alpha k^{1+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}} = \frac{1}{k^{1+\frac{1}{\alpha}-\frac{1}{\alpha\beta}+o(1)}}. \tag{19}$$

The correction term coming from the integration outside the essential domain can be estimated as follows

$$\psi_{\text{small}}(k) < \text{const} \cdot \left(\int_{-k}^{-k^{1/2+\gamma}} \frac{e^{-\frac{\varepsilon^2}{2k}+o(1)}}{(2\pi k)^{\frac{1}{2}}}\, d\varepsilon + \int_{k^{1/2+\gamma}}^{\infty} \frac{e^{-\frac{\varepsilon^2}{2k}+o(1)}}{(2\pi k)^{\frac{1}{2}}}\, d\varepsilon \right)$$

$$= o\left(\frac{1}{k^{1+\frac{1}{\alpha}-\frac{1}{\alpha\beta}}}\right). \tag{20}$$

In the case when $\varphi(\omega)$ is a power law in our weaker sense, say $\varphi(\omega) = \frac{1}{\omega^{\beta+o(1)}}$, we need some regularity condition of the form $D\varphi(\omega) = \frac{-1}{\omega^{1+\beta+o(1)}}$ on the derivative to get an asymptotic power law for $\psi(k)$.

   We continue now with the case when $\varphi(\omega)$ decays faster then any power law, that is we assume

$$\varphi(\omega) < \frac{1}{\omega^l} \qquad \text{for any} \quad l \quad \text{and} \quad \omega > \omega_0(l). \tag{21}$$

To get the desired result we have to show that formula (15) holds. Since $\varphi^{-1}(\frac{1}{\omega})$ goes to infinity for $\omega \to \infty$ we have to show

$$\frac{-\varphi(x)}{D\varphi(x)} = [\varphi(x)]^{o_x(1)}. \tag{22}$$

The last formula states that the negative logarithmic derivative of φ should not become to large or to small compared to φ respectively $\frac{1}{\varphi}$. For the following it is convenient to set $\varphi(x) = e^{-g(x)}$ with $g(x) \to \infty$ and rewrite formula (22) as

$$e^{-\mu g(x)} < \frac{1}{Dg(x)} < e^{\mu g(x)} \qquad \text{for} \quad \mu \in (0, \mu_0) \quad \text{and} \quad x > x_0(\mu). \tag{23}$$

Assume that formula (23) is not true with respect to the right hand side. Than we have for a sequence of values $\{x_i\}$ and open intervals I_i around the x_i and some function $a(x)$

$$\frac{1}{Dg(x)} = e^{\mu g(x)} a(x) \qquad \text{and} \qquad a(x) > 1 \qquad \text{for} \quad x \in I_i. \tag{24}$$

Integrating the last equation gives

$$e^{\mu g(x)} = e^{\mu g(x_0)} + \mu \int_{x_0}^x \frac{1}{a(z)} \, dz. \tag{25}$$

Since our assumption on $D[\frac{1}{\varphi(\omega)}]^\mu$ to be monotone for $\mu > 0$ and $\omega > \omega_0(\mu)$ implies $a(x) > 1$ eventually we conclude that

$$e^{\mu g(x)} < e^{\mu g(x_0)} + \mu(x - x_0). \tag{26}$$

But the fast decay condition for $\varphi(x)$ translates into a growth condition for $g(x)$ namely for all k

$$g(x) > k \log x; \qquad x > x_0(k) \tag{27}$$

which clearly contradicts formula (26). It remains to show that the left hand side of formula (23) also holds. Assuming the converse we get

$$\frac{1}{Dg(x)} = e^{-\mu g(x)} \frac{1}{a(x)} \qquad \text{and} \qquad a(x) > 1 \qquad \text{for} \quad x \in I_i \qquad (28)$$

and after integration

$$e^{-\mu g(x)} = e^{-\mu g(x_0)} - \mu \int_{x_0}^{x} a(z)\, dz. \qquad (29)$$

The monotonicity condition again implies $a(x) > 1$ eventually, hence

$$e^{-\mu g(x)} < e^{-\mu g(x_0)} - \mu(x - x_0) \qquad (30)$$

and a clear contradiction since the right hand side becomes negative for large values of x. With this result in hand we get for the leading term

$$\psi_{\text{ess}}(k) = \frac{(k_0 A)^{\frac{1}{\alpha}}(1 + o(1))}{\alpha k^{1+\frac{1}{\alpha}}} \cdot \int_{-k^{1/2+\gamma}}^{k^{1/2+\gamma}} \left(\frac{k_0 A}{k+\varepsilon}\right)^{\frac{o(1)}{\alpha}} \cdot \frac{e^{-\frac{\varepsilon^2}{2k}+o(1)}}{(2\pi k)^{\frac{1}{2}}}\, d\varepsilon$$

$$= \frac{1}{k^{1+\frac{1}{\alpha}+o(1)}} \int_{-k^{1/2+\gamma}}^{k^{1/2+\gamma}} \frac{e^{-\frac{\varepsilon^2}{2k}+o(1)}}{(2\pi k)^{\frac{1}{2}}}\, d\varepsilon = \frac{1}{k^{1+\frac{1}{\alpha}+o(1)}}. \qquad (31)$$

For the minor integrals we get

$$\psi_{\text{small}}^{(1)}(k) = \int_{-k}^{-k^{1/2+\gamma}} \frac{(k_0 A)^{\frac{1}{\alpha}}}{\alpha(k+\varepsilon)^{1+\frac{1}{\alpha}}} \cdot \frac{-\varphi \circ \varphi^{-1}\left(\left(\frac{k_0 A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right)}{(D\varphi)\left(\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right)\right)} \cdot \frac{e^{-\frac{\varepsilon^2}{2k}+o(1)}}{(2\pi k)^{\frac{1}{2}}}\, d\varepsilon \qquad (32)$$

and

$$\psi_{\text{small}}^{(2)}(k) = \int_{k^{1/2+\gamma}}^{\infty} \frac{(k_0 A)^{\frac{1}{\alpha}}}{\alpha(k+\varepsilon)^{1+\frac{1}{\alpha}}} \cdot \left(\frac{k_0 A}{k+\varepsilon}\right)^{\frac{o(1)}{\alpha}} \cdot \frac{e^{-\frac{\varepsilon^2}{2k}+o(1)}}{(2\pi k)^{\frac{1}{2}}}\, d\varepsilon$$

$$= o\left(\frac{1}{k^{1+\frac{1}{\alpha}}}\right). \qquad (33)$$

To see that $\psi_{\text{small}}^{(1)}(k)$ is of order $o(\frac{1}{k^{1+(1/\alpha)}})$ fix $\gamma > \delta > 0$ sufficiently small and some k' such that

$$\frac{-\varphi \circ \varphi^{-1}\left(\left(\frac{k_0 A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right)}{(D\varphi)\left(\varphi^{-1}\left(\left(\frac{k_0 \cdot A}{k+\varepsilon}\right)^{\frac{1}{\alpha}}\right)\right)} < \left(\frac{k}{k_0 A}\right)^{\frac{\delta}{\alpha}} \qquad \text{for} \quad k > k'. \qquad (34)$$

With

$$C_{\max}(\delta_0) = \max_{0 < k < k'} \left\{ \frac{-\varphi \circ \varphi^{-1}\left(\left(\frac{k_0 A}{k}\right)^{\frac{1}{\alpha}}\right)}{(D\varphi)\left(\varphi^{-1}\left(\left(\frac{k_0 A}{k}\right)^{\frac{1}{\alpha}}\right)\right)} \right\} < \infty \tag{35}$$

we obtain for $\psi_{\text{small}}^{(1)}(k)$ the following estimation:

$$\begin{aligned}
\psi_{\text{small}}^{(1)}(k) &< (k_0 A)^{\frac{1}{\alpha}} \cdot C_{\max}(\delta)\, k' e^{-(\frac{k}{2} - 2k')} \\
&\quad + \int_{k'}^{k - k^{1/2+\gamma}} \frac{(k_0 A)^{\frac{1}{\alpha}}}{\alpha \varepsilon^{1 + \frac{1}{\alpha}}} \cdot \left(\frac{\varepsilon}{k_0 A}\right)^{\frac{\delta}{\alpha}} \cdot \frac{e^{-\frac{k^{2\gamma}}{2} + o(1)}}{(2\pi k)^{\frac{1}{2}}}\, d\varepsilon \\
&= o\left(\frac{1}{k^{1 + \frac{1}{\alpha}}}\right).
\end{aligned} \tag{36}$$

Therefore we obtain in the case when φ decays faster then any power law that the degree distribution $\psi(k) = \frac{1}{k^{1 + (1/\alpha) + o(1)}}$ is independent of φ. ∎

2.2. The Case of Nonconstant Outdegree

Next we deal with the case when the outdegree distribution is not constant. We will always assume that the expected outdegree is finite to keep the total number of edges proportional to N. First we show that the indegree distribution is the same as in the constant degree case with k_0 replaced by $\mathbb{E}(d_{\text{out}}(x))$. With respect to the total degree distribution two situations can appear. Either the r.v. d_{out} is independent of the r.v. ω or not. In the independent case one has to take then just the convolution of indegree and outdegree distribution to get the total degree distribution. The guiding principle here is, that the dominating distribution wins. If things are not independent the analysis becomes quite more involved and we will sketch only the relevant steps dealing with a detailed investigation in a separate paper.

We start with the independent case. Let

$$q_k = \Pr\{d_{\text{out}}(x) = k\} \tag{37}$$

and

$$p_k = \Pr\{d_{\text{in}}(x) = k\} \tag{38}$$

the asymptotic outdegree respectively indegree distributions. Since we assumed asymptotic independence we get for the total degree distribution d_k the convolution

$$d_k = \sum_{0 \leqslant i \leqslant k} q_i \cdot p_{k-i}. \tag{39}$$

We say that the tail of a distribution $\{p_i\}$ dominates the tail of another distribution $\{q_i\}$ if $q_i = o(p_i)$. For the $\{p_i\}$ we assume further that it is an asymptotic power-law distribution with exponent $\gamma > 2$. We estimate d_k as follows:

$$
\begin{aligned}
d_k &= \sum_{0 \leqslant i \leqslant \frac{k}{2}} q_i \cdot p_{k-i} + \sum_{0 \leqslant i < \frac{k}{2}} q_{k-i} \cdot p_i \\
&= \sum_{0 \leqslant i \leqslant \frac{k}{2}} q_i \cdot \frac{1}{(k-i)^{\gamma+o(1)}} + \sum_{0 \leqslant i < \frac{k}{2}} o\left(\frac{1}{(k-i)^{\gamma+o(1)}}\right) \cdot \frac{1}{i^{\gamma+o(1)}} \\
&= \frac{1}{k^{\gamma+o(1)}} \sum_{0 \leqslant i \leqslant \frac{k}{2}} q_i \cdot \frac{1}{(1-\frac{i}{k})^{\gamma+o(1)}} + o\left(\frac{1}{k^{\gamma+o(1)}}\right) \sum_{0 \leqslant i < \frac{k}{2}} \frac{1}{(1-\frac{i}{k})^{\gamma+o(1)}} \cdot p_i.
\end{aligned}
\tag{40}
$$

Since both sums in the last expression are bounded and larger zero we obtain

$$d_k = \frac{1}{k^{\gamma+o(1)}}. \tag{41}$$

In the case when the outdegree distribution is the dominating one (and in our case it has to be a power-law distribution) we can just reverse the above argument to show that the total degree distribution is in this situation the outdegree distribution.

It remains to show that a variable outdegree will not affect the structure of the indegree distribution. The main difference in comparison to the previous section is that formula (7) gets replaced by

$$
\begin{aligned}
\psi_\omega(k) = \sum_M &\Pr\{\# \text{ of edges in } V_N = M\} \\
&\times \binom{M}{k} \left(\frac{A(\xi,N)}{[\varphi(\omega)]^\alpha N}\right)^k \left(1 - \frac{A(\xi,N)}{[\varphi(\omega)]^\alpha N}\right)^{M-k}.
\end{aligned}
\tag{42}
$$

Since for $M = \text{const} \cdot N$ each binomial term still converges to a Poisson distribution with parameter $\frac{\text{const} \cdot A}{[\varphi(\omega)]^\alpha}$ it is not difficult to show that in case when a limiting outdegree distribution exists and the first moment is finite one still obtains a limiting Poisson distribution where k_0 is replaced by $\mathbb{E}(d_{\text{out}})$.

2.3. The Degree Correlation and the Clustering Coefficient

It is interesting to note that graphs formed by the "Cameo Principle" show a degree affinity which is observed in many real networks and which is also one of the underlying assumption in the Albert Barabasi model, namely

$$\Pr\{\text{an edge starting in } x \text{ ends up in a vertex } y \text{ with } d(y) = k\} = \frac{k}{N} \cdot \text{const.}$$
(43)

To see that this is the case recall from the previous sections that the typical vertices with degree k have ω-values from $I_{\text{ess}}(k)$ for large k. Due to the definition of the directed edge probability we get for $|\varepsilon| < k^{\frac{1}{2}+\gamma}$

$$\Pr\{x \to y \mid \omega(y) \in I_{\text{ess}}(k)\} = \frac{A}{\left[\varphi\left(\varphi^{-1}\left(\left(\frac{k_0 A}{k+\varepsilon}\right)^{\frac{1}{\alpha}} \right) \right) \right]^{\alpha}} \cdot \frac{1}{N}$$

$$= \frac{A}{\left(\frac{k_0 A}{k+\varepsilon}\right)} \cdot \frac{1}{N} = \frac{k(1+o(1))}{k_0 N} = \frac{k(1+o(1))}{\sum_{x \in V_N} d(x)}$$

$$= \Pr\{x \to y \mid d(y) = k\}.$$
(44)

Therefore it seems to us, that if one observes in real networks edge affinities proportional to the degree one cannot conclude that the generation of the graph is also due to an edge formation process build up from this proportionality (like in the Albert–Barabasi model).

We want to close this section with a short discussion of the clustering coefficient. Since no explicit local clustering rule was implemented in our model one cannot expect a high clustering coefficient. Nevertheless it will be slightly larger then in the classical $G(N, p)$ case. Again we will study here only the situation for constant outdegree k_0. Remember that the clustering coefficient of a vertex x with degree k is defined as

$$C(x) = \frac{2 \cdot \#\{(y, z); y \sim z \text{ and } y, z \in N_1(x)\}}{k(k-1)}$$
(45)

where $N_1(x)$ is the set of neighbors of x. The global clustering coefficient is then just the expectation $\mathbb{E}(C(x))$. In what follows we will only give an explicit computation for the number of edges in $N_1(x)$ which is a little bit

simpler to compute. Since the edge formation probabilities depend only on the "target" vertices, we get on $N_1(x)$ the same ω-distribution as on the whole graph. In other words for any fixed x the probability that a $y \in N_1(x)$ has $\omega(y) > \omega_0$ is given by $\int_{\omega_0}^{\infty} \varphi(\omega) \, d\omega$. Therefore to compute $\mathbb{E}(\#\{(y, z); y \sim z \text{ and } y, z \in N_1(x)\} \mid d(x) = k)$ it is enough to compute the expected number of edges in a random vertex sample of given size k. It is easy to see that this number equals $\frac{k_0(k-1)}{N}$. Since we know further that in the limit $N \to \infty$ the degree distribution for an x with $\omega(x) = \omega$ is the Poisson distribution $\frac{c_\omega^{k-k_0}}{(k-k_0)!} e^{-c_\omega}$ with $c_\omega = \frac{k_0 \cdot A}{[\varphi(\omega)]^\alpha}$ and $k \geqslant k_0$ we get with $T(\omega) := \mathbb{E}(\#\{(y, z); y \sim z \text{ and } y, z \in N_1(x)\} \mid \omega(x) = \omega)$

$$T(\omega) = \sum_{k \geqslant k_0} \frac{k_0(k-1) \, c_\omega^{k-k_0}}{N(k-k_0)!} \, e^{-c_\omega}$$

$$= \sum_{m \geqslant 0} \frac{k_0(m+k_0-1) \, c_\omega^m}{N \cdot m!} \, e^{-c_\omega}$$

$$= \frac{k_0 c_\omega + k_0(k_0-1)}{N} \tag{46}$$

and finally

$$\int_\omega T(\omega) \, d\omega = \int \frac{k_0 \dfrac{k_0 \cdot A}{[\varphi(\omega)]^\alpha} + k_0(k_0-1)}{N} \, d\omega \tag{47}$$

as the expected number of edges among the neighbors of a random chosen vertex x. Completely analogous one can obtains the asymptotics $\frac{\text{const}}{N}$ for the expected clustering coefficient C. The explicit expression for the constant there is more cumbersome but straightforward.

3. A GROWING NETWORK MODEL

In the above sections we described a model where the total number of vertices is large but fixed and all edges are generated at once simultaneously. Since real networks are typically evolving networks many models allow a growth process for the graphs. This is easy to incorporate in our situation too. For simplicity and since it is rather common today we use the evolution concept from Albert and Barabasi.[2]

We start with an initial graph G_0 with N_0 vertices and at each time $t \in \mathbb{N}$ a fixed number L_0 of new vertices with ω-values taken again i.i.d.

with density $\varphi(\omega)$. Each vertex forms at the time when it enters the graph, k_0 edges with the vertices already present. The edges are formed according to the "Cameo Principle" that is

$$p_\omega(t) := \Pr\{x \to y \mid \omega = \omega(y) \text{ and } x \text{ enters the graph at time } t\}$$
$$= \frac{A(\xi_t, N_t)}{[\varphi(\omega)]^\alpha} \cdot \frac{1}{N_t} \tag{48}$$

where ξ_t is the realization of the ω-values at time t, $N_t = tL_0 + N_0$, and $A(\xi_t, N_t)$ a normalization constant. Let $d(x, t, \tau)$ be the degree of x at time t and $\tau(x) = \tau$ the time when x entered the graph. For x fixed and $t \gg \tau$ we have

$$\mathbb{E}(d_{\text{in}}(x, t)) = \sum_{i > \tau}^{t} \frac{k_0 A(\xi_i, N_i)}{[\varphi(\omega(x))]^\alpha} \cdot \frac{1}{N_t} \simeq \frac{k_0 A}{[\varphi(\omega(x))]^\alpha \cdot L_0} (\log t - \log \tau). \tag{49}$$

We now investigate the asymptotic indegree distribution, namely the asymptotic probability that a randomly chosen vertex from the evolving graph at time t has indegree k. For simplicity we deal here only with the case $L_0 = 1$. We first look at the indegree distribution $\psi_\omega(k, \tau, t) = \Pr\{d_{\text{in}}(x, t) = k\}$ for a fixed x with $\omega = \omega(x)$ and $\tau = \tau(x)$. Since

$$\psi_\omega(k, \tau, t) = \Pr\left\{ \sum_{y:\, \tau(y) > \tau(x)} \mathbf{1}_{y \sim x} = k \right\} \tag{50}$$

with $\mathbf{1}_{y \sim x}$ being the independent r.v. of the indicator function that y formed an edge with x, one gets by a well known theorem in probability theory for $t - \tau$ large a limiting Poisson distribution with expectation equal to $\mathbb{E}(d_{\text{in}}(x, t))$ (see ref. 6 for recent sharp results in this direction). In complete analogy to Section 2 we obtain after integration over ω a scale free distribution for $\psi(k, \tau, t)$—the not on ω conditioned indegree distribution. Taking explicit care on the τ dependence we get

$$\psi(k, \tau, t) = \frac{(k_0 A[\log t - \log \tau])^{\frac{1}{\alpha}}}{k^{1 + \frac{1}{\alpha} + o(1)}}; \qquad t - \tau > \frac{k}{k_0}. \tag{51}$$

Note that $o(1)$ does not depend on τ for $t - \tau \gg k$. To obtain the final (not on τ conditioned) distribution $\psi(k, t)$ we just have to sum

$$\psi(k, t) = \frac{1}{t} \sum_{0 < \tau < t} \psi(k, \tau, t)$$

$$= \frac{1}{t} \sum_{0 < \tau < t} \frac{(k_0 A[\log t - \log \tau])^{\frac{1}{\alpha}}}{k^{1+\frac{1}{\alpha}+o(1)}}$$

$$= \frac{1}{k^{1+\frac{1}{\alpha}+o(1)}}. \tag{52}$$

In the last two equalities we have implicitly used the fact that the terms with $t - \tau \sim \text{const} \cdot k$ do not really contribute to the $\psi(k, t)$ distribution and that $\sum_{0 < \tau < t} \log \tau \sim t \log t - t$.

4. A MODEL VARIANT WITH HIGH CLUSTERING COEFFICIENT (THE ''MY FRIENDS ARE YOUR FRIENDS''-PRINCIPLE)

To get high clustering coefficients one has to incorporate local edge formation rules into the model like in the famous small world model of Watts and Strogatz.[7] A natural way to do this for our graphs is by introducing a kind of additional local choice rule: with probability p make the choice into the neighbors of your own neighbors at the given time and with probability $q = 1 - p$ proceed like above. We call this rule the ''my fiends are your friends'' principle for obvious reasons. It should be a well known principle in social sciences since many social contacts are formed that way. We will give a detailed investigation of the types of graphs resulting from such processes in a forthcoming paper.[4] Here we will present only an informal discussion of some of the main properties. First it is important to note that the order in which the edges are created plays now a role (at least for the fine structure of local properties of the graphs). To fix the setting let us consider the following rule. The enumeration of the vertex set we will associate with the time at which the vertex will make a choice. The first choice each vertex makes is an independent one according to the rules specified in the previous paragraph. For the next choices any vertex x has two options—with probability $q = 1 - p$ he proceeds as with the first choice (independent, global choice) or with probability p he chooses from $N_2(x, t)$, the set of vertices at distance 2 from x at time t (dependent, local choice). We remark that the concrete form of $\varphi(\omega)$ is irrelevant for the further discussion. Two situations are now natural to consider:

Case A. At an initial set of choices all vertices make their first, independent choice (here the order in which the choices are made does not matter). After that, each vertex makes it's second choice at the time corre-

sponding to it's index number, then all vertices make their third choice and so on (to avoid a bias one should renumber randomly the vertex set after each sequence of choices).

Case B. Each vertex makes the remaining $k_0 - 1$ choices immediately after the first one. Again the time when a vertex is about to make the choices corresponds to it's index number.

Let $T(x)$ be the number of triangles in the final graph containing x. There are some trivial lower and upper bounds on the expectation of $T(x)$. First observe that due to the bounded outdegree there cannot be more than $(k_0 - 1) d(x)$ edges among $N_1(x)$. On the other side any choice made by x according to the "my friends are your friends"-rule automatically generates a triangle containing x. Therefore $\mathbb{E}(T(x)) \geqslant p(k_0 - 1)$ and hence the correlation coefficient is independent of N asymptotically. We claim that for large $d(x)$ the conditional expectation given the degree is approximately $\mathbb{E}(T(x) \mid d(x)) \simeq \text{const} \cdot d(x)$. We outline the reason for that for $k_0 = 2$ and the situation in Case B. Note that the rule from Case B is also the canonical one if one wants to deal with growing networks.

Let $d_{\text{in}}^1(x)$ be the indegree of x after each vertex made the first choice. Denote further by $N_l^1(x)$ the set of vertices at distance l from x after the first choices are made. Since the first choices are independent, a vertex x is not part of a triangle almost surely. In the second round of choices triangular configurations will be formed frequently. Since we assumed that $d(x)$ is large we can assume that $d_{\text{in}}^1(x)$ is large too. There are two mechanisms which contribute to $T(x)$ when the second choices are made. First all vertices in $N_l^1(x)$ create an edge into $N_l^1(x)$ with probability p and since $|N_1^1(x)| = d_{\text{in}}^1(x) + 1$ is assumed to be large we can neglect the events that multiple edges are created. So we expect $p(d_{\text{in}}^1(x) + 1)$ triangles containing x from this process. Second any vertex z from $N_2^1(x)$ can chose x with probability $\frac{p}{|N_2^1(z)|}$ and therefore create a new triangle. Furthermore for most vertices $z \in N_2^1(x)$ we have

$$|N_2^1(x)| \sim d_{\text{in}}^1(x) \cdot \text{const} \cdot |N_2^1(z)| \tag{53}$$

from which we get an estimation of order $pd_{\text{in}}^1(x) \cdot \text{const}$ for the expected number of triangles generated via this process. Combining these estimations and using the fact that the number of independent choices pointing to x in the second choice sequence is about $(1 - p) d_{\text{in}}^1(x)$ we obtain

$$d(x) \sim d_{\text{in}}^1(x) + (1 - p) d_{\text{in}}^1(x) + p(d_{\text{in}}^1(x) + 1) + pd_{\text{in}}^1(x) \cdot \text{const}$$
$$\sim d_{\text{in}}^1(x)(2 + p \cdot \text{const}) \tag{54}$$

which gives for the expected number of triangles

$$\mathbb{E}(T(x) \mid d(x)) \sim \frac{p(1+\text{const})}{2+p \cdot \text{const}} d(x). \qquad (55)$$

The above considerations are of course of heuristic type but can easily made be precise. The const appearing in the above formulas can as well be explicitly computed. We remark finally that the above considerations remain true if the outdegree distribution is not a constant and that the clustering mechanism does not change the asymptotic shape of the total degree distribution. We plan to discuss all this aspects in much more detail in a paper in preparation.

5. CONCLUSIONS AND OUTLOOK

We have shown in this article that graphs with a scale-free degree distribution appear naturally as the result of a simple edge formation rule based on choices where the probability to chose a vertex with affinity parameter ω is proportional to the frequency of appearance of that parameter. If the affinity parameter ω is itself power law like distributed one could also use a direct proportionality to the value ω to get still a scale free graph since $\varphi(\omega)$ is itself a polynomial relation (this explains the observation reported in ref. 5).

We have not tried in this paper to give estimations on the error terms for the asymptotics. Numerical simulations, which will be reported elsewhere, indicate that already for values of k of order $\text{const} \cdot k_0$ and vertex sets of size $N \approx 10^5$ one has a very good agreement with the asymptotic result. Of course in all these finite size effects the concrete form of φ really matters.

It seems natural to investigate models where $\varphi(\omega)$ has a singular, for instance discrete, support. Such a situation is common when one couples for instance the choice probabilities directly to the outdegree (see ref. 3 for a specific model of such type). Some care is necessary in that case. For instance if one takes a discrete integer valued random variable k with the distribution $\varphi(k) = \text{const} \cdot \exp(-k)$ and proceeds like in the case of the continuous variable ω it will be not longer true that the asymptotic degree distribution is a power-law distribution. The reason behind this is the following. The expected indegree of vertices x and y with $k(x) = k_0$ and $k(y) = k_0 + 1$ differs by a constant factor but since the indegree of x and y is concentrated in an interval of length square root of the expectation around the expectation there appears a gap in the degree distribution of the whole graph. If $\varphi(k)$ is a power law distribution with exponent β the same

remains true for $\alpha > \frac{2}{\beta}$ whereas for $\alpha < \frac{2}{\beta}$ a similar proof with the same results as in the continuous case can be given. In general one needs for singular $\varphi(\omega)$ an overlap condition for the involved Poisson distributions to get a power law tail distribution.

With respect to the clustering mechanism described in Section 4 there are many interesting questions. What is the effect on diameter, average path-length and spectral properties?

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